

Poincaré isomorphism in K -theory on manifolds with edges

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Abstract

The aim of this paper is to construct the Poincaré isomorphism in K -theory on manifolds with edges. We show that the Poincaré isomorphism can naturally be constructed in the framework of noncommutative geometry. More precisely, to a manifold with edges we assign a noncommutative algebra and construct an isomorphism between the K -group of this algebra and the K -homology group of the manifold with edges viewed as a compact topological space.

1 Introduction

Let M be a smooth closed even-dimensional manifold equipped with a complex spin structure (spin^c -structure in what follows). Then in K -theory we have the Poincaré isomorphism

$$\tau: K^0(M) \longrightarrow K_0(M) \tag{1}$$

between the K -group of vector bundles on the manifold and the K -homology group [1, 2, 3]. From the viewpoint of analysis (to which we stick here), the K -homology group is naturally identified with the group generated by elliptic operators on M . In this language, the mapping (1) takes the class of a vector bundle on M to the class of the spin^c Dirac operator twisted by the vector bundle.

The aim of this paper is to construct a Poincaré isomorphism similar to (1) for the case of manifolds with edges. Manifolds with edges are a class of manifolds with nonisolated singularities. In this situation, we show that the Poincaré isomorphism can naturally be constructed in the framework of noncommutative geometry. More precisely, to a manifold \mathcal{M} with edges we assign a noncommutative algebra \mathcal{A} and obtain an isomorphism

$$K_0(\mathcal{A}) \longrightarrow K^0(C(\mathcal{M})) \tag{2}$$

between the K -group of \mathcal{A} and the analytic K -homology group of the algebra $C(\mathcal{M})$ of continuous functions on the manifold with edges viewed as a compact topological space. If \mathcal{M} is a smooth manifold, then \mathcal{A} is just $C(\mathcal{M})$ and (2) is none other than (1) (the K -groups of spaces being identified with the K -groups of the corresponding function algebras).

The main difficulty in constructing the isomorphism (2) is that one has to find a realization of the Dirac operator on the smooth part of \mathcal{M} as an elliptic operator over the algebra $C(\mathcal{M})$ of continuous functions on \mathcal{M} . It turns out that this is not always possible. The corresponding obstruction was computed in [4] and is a generalization of the Atiyah–Bott obstruction [5] in the theory of classical boundary value problems. For the case in which the obstruction vanishes, we construct a Fredholm realization of the Dirac operator in the class of boundary value problems introduced in [6].

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2 Geometry

Manifolds with edges. Let M be a compact smooth manifold with boundary ∂M , and suppose that ∂M is the total space of a smooth locally trivial bundle

$$\pi: \partial M \longrightarrow X$$

over a closed smooth base X with fiber $\pi^{-1}(x) \equiv \Omega_x$ over $x \in X$, which is a smooth closed manifold. For simplicity, we assume that X and Ω_x are connected.

Definition 2.1. The topological space obtained from M by identifying points in each fiber of π is called the *manifold with edge* corresponding to the pair (M, π) . Let \mathcal{M} denote the manifold with edge.

In this paper, we construct Poincaré duality for the case in which M and X are even-dimensional manifolds (and hence the fiber Ω is odd-dimensional automatically).

1. Riemannian metric. We fix a Riemannian metric on M and assume that it is of product type

$$g_M = dt^2 + g_{\partial M}$$

in a collar neighborhood $\partial M \times [0, 1]$ of the boundary, where $t \in [0, 1]$ is the coordinate normal to the boundary and $g_{\partial M}$ is a metric on the boundary compatible with π (a so-called *submersion metric*). Namely, a connection in π gives the direct sum decomposition

$$T\partial M \simeq T^H\partial M \oplus T^V\partial M \simeq \pi^*TX \oplus T\Omega \quad (3)$$

of the tangent bundle into horizontal and vertical components. The vertical component coincides with the tangent bundle to the fibers, and the horizontal component is naturally isomorphic to π^*TX . Using the decomposition (3), one defines a submersion metric on the boundary as

$$g_{\partial M} = \pi^*g_X + g_{\Omega}, \quad (4)$$

where g_X is a metric on X and g_{Ω} is a family of metrics on the fibers Ω .

For the cotangent bundle $T^*\partial M$, there is a similar decomposition

$$T^*\partial M \simeq \pi^*T^*X \oplus T^*\Omega. \quad (5)$$

Recall how the decompositions (3) and (5) are defined. Let

$$\omega: T\partial M \rightarrow T\Omega, \quad \omega|_{T\Omega} = Id,$$

be a connection in π (a projection onto the vertical subbundle). Then (3) is, by definition,

$$T\partial M \simeq \ker \omega \oplus T\Omega.$$

The isomorphism $\ker \omega \xrightarrow{\sim} \pi^*TX$ is induced by the projection $\pi_*: T\partial M \rightarrow TX$.

In a similar way, (5) is defined as

$$T^*\partial M \simeq \pi^*T^*X \oplus (\ker \omega)^\perp,$$

where the fiber of the subbundle $(\ker \omega)^\perp$ is, by definition, the space of 1-forms annihilating the horizontal subspace $\ker \omega$. The isomorphism $(\ker \omega)^\perp \rightarrow T^*\Omega$ is induced by restriction of forms to the vertical subbundle $T\Omega \subset T\partial M$.

2. spin^c-structure and the Dirac operator. Suppose that both the base X and the fibers Ω are equipped with spin^c-structures. These data induce a spin^c-structure on ∂M .¹ We shall also assume that M is equipped with a spin^c-structure compatible with that on ∂M .

Then we can define the spin^c Dirac operator

$$D_M: C^\infty(M, S_M^+) \longrightarrow C^\infty(M, S_M^-), \quad S_M \simeq S_M^+ \oplus S_M^-,$$

on M . This operator acts on sections of spinor bundles $S_M^\pm \in \text{Vect}(M)$ (e.g., see [8]).

3. The principal symbol of the Dirac operator. In this paper, we shall only deal with the (principal) symbol of the Dirac operator. Let us describe it explicitly. First, by our choice of the direct product metric near the boundary,

$$D_M \Big|_{\partial M \times [0,1]} \simeq \frac{\partial}{\partial t} + D_{\partial M}, \quad (6)$$

where $D_{\partial M}: C^\infty(\partial M, S_{\partial M}) \longrightarrow C^\infty(\partial M, S_{\partial M})$ is the Dirac operator on the boundary.

Second, the spinor bundle of the boundary has the decomposition (see [9])

$$S_{\partial M} \simeq \pi^*S_X \otimes S_\Omega = \pi^*(S_X^+ \oplus S_X^-) \otimes S_\Omega, \quad (7)$$

where $S_X \simeq S_X^+ \oplus S_X^- \in \text{Vect}(X)$ is the spinor bundle of the base and $S_\Omega \in \text{Vect}(\partial M)$ is the spinor bundle of the fibers.

¹For our purposes, one can use the following definition (cf. [7]): a spin^c-structure on M is a bundle S_M of irreducible modules over the bundle $Cl(M)$ of Clifford algebras on M . Here we assume that some Riemannian metric on M is given. If M is even-dimensional and oriented, then $S_M \simeq S_M^+ \oplus S_M^-$.

In this notation, the symbol of $D_{\partial M}$ is given by

$$\sigma(D_{\partial M})(\xi, \eta) = \begin{pmatrix} 1 \otimes c_{\Omega}(\eta) & c_X(\xi) \otimes 1 \\ c_X(\xi) \otimes 1 & -1 \otimes c_{\Omega}(\eta) \end{pmatrix} : \begin{array}{c} \pi^* S_X^+ \otimes S_{\Omega} \\ \oplus \\ \pi^* S_X^- \otimes S_{\Omega} \end{array} \longrightarrow \begin{array}{c} \pi^* S_X^+ \otimes S_{\Omega} \\ \oplus \\ \pi^* S_X^- \otimes S_{\Omega} \end{array}, \quad (8)$$

where for $T^* \partial M$ we use the decomposition (5) and write

$$(\xi, \eta) \in \pi^* T^* X \oplus T^* \Omega \simeq T^* \partial M$$

and

$$\begin{aligned} c_X : T^* X \otimes S_X &\longrightarrow S_X, \quad c_{\Omega} : T^* \Omega \otimes S_{\Omega} \longrightarrow S_{\Omega}, \\ c_X(\xi) c_X(\xi') + c_X(\xi') c_X(\xi) &= 2(\xi, \xi')_{g_X}, \\ c_{\Omega}(\eta) c_{\Omega}(\eta') + c_{\Omega}(\eta') c_{\Omega}(\eta) &= 2(\eta, \eta')_{g_{\Omega}}, \end{aligned}$$

are Clifford multiplications.

Let Q_0 be the positive spectral projection of $D_{\partial M}$. The operator Q_0 is a pseudodifferential operator of order zero on ∂M . Its symbol is equal to the positive spectral projection of $\sigma(D_{\partial M})$ and hence is given by

$$\sigma(Q_0)(\xi, \eta) = \frac{1}{2} \begin{pmatrix} 1 \otimes (1 + c_{\Omega}(\eta)) & c_X(\xi) \otimes 1 \\ c_X(\xi) \otimes 1 & 1 \otimes (1 - c_{\Omega}(\eta)) \end{pmatrix} \quad \text{for } |\xi|^2 + |\eta|^2 = 1. \quad (9)$$

(This follows from (8) in view of the fact that the square of the symbol (8) is a scalar symbol.)

3 Boundary value problems for twisted Dirac operators

The fibers of $\pi : \partial M \rightarrow X$ are odd-dimensional spin^c-manifolds. Hence we can consider the index (e.g., see [10])

$$\text{ind } D_{\Omega} \in K^1(X)$$

of the family, parametrized by X , of self-adjoint Dirac operators in the fibers. We shall assume that the following condition is satisfied.

Assumption 3.1.

$$\text{ind } D_{\Omega} = 0.$$

It will be shown below (see Proposition 4.3) that this condition is necessary for the Dirac operator to define a class in the K -homology of the manifold with edges.

If Assumption 3.1 is satisfied, then one can make the family D_{Ω} invertible by perturbing it by a smooth family of finite rank operators [11, Prop. 1]. Doing so if necessary, we assume from now on that the family D_{Ω} is invertible. Then the family of positive spectral projections

$$\Pi_+ = \Pi_+(D_{\Omega}) : C^{\infty}(\Omega, S_{\Omega}) \longrightarrow C^{\infty}(\Omega, S_{\Omega})$$

of the Dirac operators in the fibers is smooth. Let $\Pi_- = 1 - \Pi_+$ be the family of complementary projections.

1. The algebra dual to the algebra of functions on a manifold with edges. Let $\mathcal{T}(\Omega)$ denote the bundle of algebras over X whose fiber

$$\mathcal{T}(\Omega_x) \subset \mathcal{B}L^2(\Omega_x, S_{\Omega_x})$$

at the point x is the algebra of zero-order Toeplitz operators on the range of the projection $\Pi_+(D_{\Omega_x})$. Namely, the elements of $\mathcal{T}(\Omega_x)$ are operators of the form

$$F = \Pi_+(D_{\Omega_x}) \tilde{F} \Pi_+(D_{\Omega_x})$$

acting on sections of $S_{\Omega_x} \in \text{Vect}(\Omega_x)$, where $\tilde{F}: C^\infty(\Omega_x, S_{\Omega_x}) \rightarrow C^\infty(\Omega_x, S_{\Omega_x})$ is a pseudodifferential operator with scalar symbol $\sigma(\tilde{F}) \in C^\infty(\Omega_x)$. (Note that $\mathcal{T}(\Omega_x)$ is a subalgebra of the algebra of classical pseudodifferential operators acting on sections of $S_{\Omega_x} \in \text{Vect}(\Omega_x)$.) By definition, we set $\sigma(F) = \sigma(\tilde{F})$ (the symbol of the Toeplitz operator F); this is, of course, different from the symbol of F as of a pseudodifferential operator.

Consider the noncommutative algebra

$$\mathcal{A} = \{(f, F) \in C^\infty(M) \oplus C^\infty(X, \mathcal{T}(\Omega)) \mid \sigma(F) = f|_{\partial M}\}. \quad (10)$$

Its elements are pairs (f, F) , where f is a function on M , F is a section of $\mathcal{T}(\Omega)$, and the restriction of f to ∂M is equal to $\sigma(F)$.

This algebra naturally acts on the space

$$C^\infty(M) \oplus C^\infty(\partial M, S_\Omega).$$

(The action on the subspace $0 \oplus \text{Im } \Pi_-(D_\Omega)$ is defined to be zero.)

Consider the norm closure $\overline{\mathcal{A}}$ of the algebra \mathcal{A} in the space of operators on $L^2(M) \oplus L^2(X, L^2(\Omega, S_\Omega))$. (Note that the operators in \mathcal{A} take the subspace $0 \oplus L^2(X, \Pi_- L^2(\Omega, S_\Omega))$ to zero.) The algebra \mathcal{A} is a local unital C^* -subalgebra of $\overline{\mathcal{A}}$. Its even K -group is generated by formal differences of projections in matrix algebras over \mathcal{A} .

To a projection²

$$\mathcal{P} = (p, P) \in \text{Mat}(n, \mathcal{A}),$$

we wish to assign a Fredholm boundary value problem and an element in the K -homology of the algebra $C(\mathcal{M})$ of continuous functions on \mathcal{M} . Here we treat $C(\mathcal{M})$ as the closure of the algebra $C^\infty(\mathcal{M}) \subset C^\infty(M)$ of smooth functions constant on the fibers of π .

²A projection \mathcal{P} is a compatible pair (p, P) of projections. The first component defines a subbundle

$$\text{Im } p \subset M \times \mathbb{C}^n,$$

and the second component defines a bundle

$$\text{Im } P \subset C^\infty(\partial M, S_\Omega) \otimes \mathbb{C}^n$$

(in general, infinite-dimensional) over X .

2. Boundary value problems for Dirac operators. Consider the Dirac operator on M twisted by the range of p viewed as a vector bundle on M :

$$D_M \otimes 1_{\text{Im } p} = (1 \otimes p)(D_M \otimes 1_{\mathbb{C}^n})(1 \otimes p): pC^\infty(M, S_M^+ \otimes \mathbb{C}^n) \longrightarrow pC^\infty(M, S_M^- \otimes \mathbb{C}^n).$$

It turns out that a compatible pair (p, \mathcal{P}) permits one to construct a well-posed boundary value problem for the twisted Dirac operator.

By $D_M \otimes 1_{\text{Im } \mathcal{P}}$ we denote the boundary value problem

$$\begin{cases} (D_M \otimes 1_{\text{Im } p})u &= f, \\ (\Delta_{\partial M}^{s/2-1/4} u_1 + v_1) + \tilde{D}_X^* v_2 &= g_1, \\ (1 \otimes (1 - P)) \left[\tilde{D}_X v_1 - (\Delta_{\partial M}^{s/2-1/4} u_2 + v_2) \right] &= g_2, \end{cases} \quad (11)$$

where $s \geq 1$ is some number and

$$u \in pC^\infty(M, S_M^+ \otimes \mathbb{C}^n), \quad f \in pC^\infty(M, S_M^- \otimes \mathbb{C}^n),$$

$$u|_{\partial M} = u_1 + u_2 \in pC^\infty(\partial M, \pi^* S_X^+ \otimes S_\Omega \otimes \mathbb{C}^n) \oplus pC^\infty(\partial M, \pi^* S_X^- \otimes S_\Omega \otimes \mathbb{C}^n),$$

$$v_1 \in (1 - P)C^\infty(\partial M, \pi^* S_X^+ \otimes S_\Omega \otimes \mathbb{C}^n), \quad v_2 \in (1 - p)C^\infty(\partial M, \pi^* S_X^- \otimes S_\Omega \otimes \mathbb{C}^n),$$

$$g_1 \in C^\infty(\partial M, \pi^* S_X^+ \otimes S_\Omega \otimes \mathbb{C}^n), \quad g_2 \in (1 - P)C^\infty(\partial M, \pi^* S_X^- \otimes S_\Omega \otimes \mathbb{C}^n).$$

Here by $\Delta_{\partial M}$ we denote Laplace type operators on ∂M acting on sections of the corresponding bundles. Let

$$\tilde{D}_X, \tilde{D}_X^*: C^\infty(\partial M, \pi^* S_X^\pm \otimes S_\Omega \otimes \mathbb{C}^n) \longrightarrow C^\infty(\partial M, \pi^* S_X^\mp \otimes S_\Omega \otimes \mathbb{C}^n)$$

be pseudodifferential operators of order zero on ∂M with principal symbol

$$c_X(\xi) \otimes 1_{S_\Omega \otimes \mathbb{C}^n}, \quad |\xi|^2 + |\eta|^2 = 1. \quad (12)$$

(This symbol is smooth on $T^* \partial M$ away from the zero section, since $c_X(\xi)$ is a linear function.)

Example 3.2. If the edge X is a point, then we can take $\mathcal{P} = (1, \Pi_+)$. (Assumption 3.1 is satisfied automatically, since $K^1(pt) = 0$.) In this case, the last equation in (11) disappears, and we obtain the boundary value problem

$$\begin{cases} D_M u &= f, \\ \Delta_{\partial M}^{s/2-1/4} u_1 + v_1 &= g_1, \end{cases}$$

where $v_1 \in (1 - \Pi_+)C^\infty(\partial M, S_\Omega)$, $g_1 \in C^\infty(\partial M, S_\Omega)$. The index of this boundary value problem is equal to the index of the Atiyah–Patodi–Singer boundary value problem [12]

$$\begin{cases} D_M u &= f, \\ \Pi_+ u|_{\partial M} &= g \in \Pi_+ C^\infty(\partial M, S_\Omega). \end{cases}$$

The unknown v_1 and the right-hand side g_2 in (11) belong to subspaces defined as the ranges of families of pseudodifferential projections in the fibers of π . In the general case, the range of such a family is not the space of sections of any vector bundle on ∂M . Therefore, our boundary value problem is not classical. However, boundary value problems of this form were studied in [6]. Let us use the finiteness theorem in the cited paper to prove that problem (11) defines a Fredholm operator in appropriate Sobolev spaces.

Proposition 3.3. *Let $s \geq 1$. Then the boundary value problem (11) defines a Fredholm operator*

$$D_M \otimes 1_{\text{Im } \mathcal{P}}: \begin{array}{c} pH^s(M, S_M^+ \otimes \mathbb{C}^n) \\ \oplus \\ (1 - P)L^2(\partial M, \pi^* S_X^+ \otimes S_\Omega \otimes \mathbb{C}^n) \end{array} \longrightarrow \begin{array}{c} pH^{s-1}(M, S_M^- \otimes \mathbb{C}^n) \\ \oplus \\ L^2(\partial M, \pi^* S_X^+ \otimes S_\Omega \otimes \mathbb{C}^n) \\ \oplus \\ (1 - P)L^2(\partial M, \pi^* S_X^- \otimes S_\Omega \otimes \mathbb{C}^n) \end{array} . \quad (13)$$

Proof. Making the standard reduction of the boundary value problem to the boundary, we see that it suffices to prove the Fredholm property of the system of equations

$$\begin{cases} (\Delta_{\partial M}^{s/2-1/4} u_1 + v_1) + \tilde{D}_X^* v_2 &= g_1, \\ (1 \otimes (1 - P)) [\tilde{D}_X v_1 - (\Delta_{\partial M}^{s/2-1/4} u_2 + v_2)] &= g_2, \end{cases} \quad (14)$$

on the boundary, where the pair (u_1, u_2) is in the range of the Calderón projection of the Dirac operator $D_M \otimes 1_{\text{Im } p}$. This Calderón projection is modulo compact operators equal to the positive spectral projection of the Dirac operator $D_{\partial M} \otimes 1_{\text{Im } p}$ on the boundary. Hence we can use the spectral projection, which we denote by Q , instead of the Calderón projection in the proof of the Fredholm property of (14). As we mentioned earlier, Q is a pseudodifferential operator with symbol $\sigma(Q) = \sigma(Q_0) \otimes 1_{\text{Im } p}$, where the symbol of Q_0 is given in (9).

The operators on ∂M occurring in (14) are special cases of operators with discontinuous symbols on fibered manifolds [6]. The Fredholm property of such operators is equivalent to the invertibility of the *principal symbol* $\sigma_{\partial M}$ on $T^* \partial M \setminus \pi^* T^* X$ and the *operator-valued symbol* σ_X defined on $T^* X \setminus 0$ and ranging in pseudodifferential operators in the fibers.

The invertibility of both symbols in our case can be proved by a straightforward computation.

A. Invertibility of the operator-valued symbol σ_X . We wish to show that the system

$$\begin{cases} u_1 + v_1 + (c_X(\xi) \otimes 1)v_2 = g_1 \\ (c_X(\xi) \otimes 1)v_1 - [1 \otimes (1 - P)](u_2 + v_2) = g_2 \end{cases}, \quad |\xi| = 1, \quad (15)$$

is uniquely solvable for all $x \in X$ and $\xi \in S_x^* X$, where

$$u = (u_1, u_2) \in \text{Im } \sigma_X(Q)(\xi), \quad \sigma_X(Q)(\xi) = \frac{1}{2} \begin{pmatrix} 1 \otimes 1 & c_X(\xi) \otimes 1 \\ c_X(\xi) \otimes 1 & 1 \otimes 1 \end{pmatrix},$$

$$\begin{aligned} v_1 &\in (S_X^+ \otimes \text{Im}(1 - P))_x, & v_2 &\in (S_X^- \otimes L^2(\Omega, S_\Omega \otimes \text{Im}(1 - p)))_x, \\ g_1 &\in (S_X^+ \otimes L^2(\Omega, S_\Omega \otimes \mathbb{C}^n))_x, & g_2 &\in (S_X^- \otimes \text{Im}(1 - P))_x. \end{aligned}$$

Note that (15) is obtained from (14) if we replace operators by their operator-valued symbols. We have also used the fact that the operator-valued symbol of a family of pseudodifferential operators in the fibers is equal to the family itself (we apply this property to P) and the operator-valued symbol of an operator with smooth symbol is equal to the restriction of the symbol to $\pi^*T^*X \subset T^*\partial M$ (we apply this property to \tilde{D}_X , \tilde{D}_X^* , and Q). Finally, note that we have used the relation $[1 \otimes P, c_X(\xi) \otimes 1] = 0$ to obtain (15) from (14).

Let us prove the triviality of the kernel. If $\sigma_X(Q)(\xi)u = u$, then $u_1 = (c_X(\xi) \otimes 1)u_2$, where u_2 is arbitrary. The first equation in (15) gives $u_1 + (c_X(\xi) \otimes 1)v_2 = -v_1$. By substituting this into the second equation, we obtain

$$v_1 - (-v_1) = 0.$$

Thus, $v_1 = 0$ and $u_1 + (c_X(\xi) \otimes 1)v_2 = 0$. Therefore, $u_1 = 0$ and $v_2 = 0$. This shows that the kernel is trivial.

Let us prove the triviality of the cokernel. First, g_2 in (15) can be assumed to be zero (by an appropriate choice of v_1). Hence we have to find the solution of the system

$$\begin{cases} u_1 + v_1 + (c_X(\xi) \otimes 1)v_2 = g \\ v_1 = [1 \otimes (1 - P)]((c_X(\xi) \otimes 1)u_2 + (c_X(\xi) \otimes 1)v_2). \end{cases}$$

By substituting the second equation into the first equation, we obtain

$$u_1 + [1 \otimes (1 - P)](u_1 + (c_X(\xi) \otimes 1)v_2) + (c_X(\xi) \otimes 1)v_2 = g,$$

or

$$[1 \otimes (2 - P)](u_1 + (c_X(\xi) \otimes 1)v_2) = g,$$

but the operator $2 - P$ is invertible (recall that P is a projection), and $u_1 + (c_X(\xi) \otimes 1)v_2$ can be arbitrary. Thus, the last equation is solvable for any g . This shows that the cokernel is trivial.

B. Invertibility of the principal symbol $\sigma_{\partial M}$. Since the dimensions of the bundles where the symbol acts are equal, it suffices to show that the corresponding homogeneous equation

$$\begin{cases} u_1 + v_1 + (c_X(\xi) \otimes 1)v_2 = 0, \\ (c_X(\xi) \otimes 1)v_1 - [1 \otimes (1 - \sigma_{\partial M}(P)(\xi, \eta))] (u_2 + v_2) = 0, \end{cases} \quad (16)$$

where

$$u = (u_1, u_2) \in \text{Im } \sigma_{\partial M}(Q)(\xi, \eta),$$

$$\sigma_{\partial M}(Q)(\xi, \eta) = \frac{1}{2} \begin{pmatrix} 1 \otimes (1 + c_\Omega(\eta)) & c_X(\xi) \otimes 1 \\ c_X(\xi) \otimes 1 & 1 \otimes (1 - c_\Omega(\eta)) \end{pmatrix},$$

$$v_1 \in (\pi^* S_X^+ \otimes \text{Im}(1 - \sigma_{\partial M}(P)))_m, \quad v_2 \in (\pi^* S_X^- \otimes S_\Omega \otimes \text{Im}(1 - p))_m,$$

$$g_1 \in (S_X^+ \otimes S_\Omega \otimes \mathbb{C}^n)_m, \quad g_2 \in (S_X^+ \otimes \text{Im}(1 - \sigma_{\partial M}(P)))_m,$$

has only the trivial solution if $m \in \partial M$, $(\xi, \eta) \in T_m^* \partial M$, $|\xi|^2 + |\eta|^2 = 1$, and $\eta \neq 0$.

Indeed, since $\sigma_{\partial M}(Q)(\xi, \eta)u = u$, we obtain

$$(1 \otimes (1 + c_\Omega(\eta)))u_1 + (c_X(\xi) \otimes 1)u_2 = 2u_1;$$

i.e., $(c_X(\xi) \otimes 1)u_2 = (1 \otimes (1 - c_\Omega(\eta)))u_1$.

A computation shows that the principal symbol of the fiberwise operator P is

$$\sigma_{\partial M}(P)(\xi, \eta) = p\sigma(\Pi_+)(\eta).$$

(See the compatibility condition in the definition of \mathcal{A} in (10).)

Let us consider two possibilities.

1. $\xi \neq 0$. By substituting the expression for $\sigma_{\partial M}(P)$ into (16), we obtain

$$\begin{cases} u_1 + v_1 + (c_X(\xi) \otimes 1)v_2 = 0, \\ |\xi|^2 v_1 - \left[1 \otimes (1 - p\sigma(\Pi_+)(\eta)) \right] \left((1 \otimes (1 - c_\Omega(\eta)))u_1 + (c_X(\xi) \otimes 1)v_2 \right) = 0. \end{cases} \quad (17)$$

The second equation of the system is equivalent to

$$v_1(1 + |\xi|^2) + \left[1 \otimes (1 - \sigma(\Pi_+)(\eta)) \right] (1 \otimes c_\Omega(\eta))u_1 = 0.$$

It is now straightforward to prove the desired triviality of the solution by using the last equation together with the first equation in (17). To this end, we decompose all vectors in the system along the ranges of the orthogonal projections

$$1 \otimes p\sigma(\Pi_+)(\eta), \quad 1 \otimes p(1 - \sigma(\Pi_+)(\eta)), \quad 1 \otimes (1 - p)$$

and match the corresponding components.

2. $\xi = 0$. In this case, (16) is reduced to

$$u_1 + v_1 = 0, \quad \left[1 \otimes (1 - p\sigma(\Pi_+)(\eta)) \right] (u_2 + v_2) = 0. \quad (18)$$

In addition, the equation $\sigma_{\partial M}(Q)(0, \eta)u = u$ implies that $u_1 \in p\text{Im}\sigma(\Pi_+)(\eta)$ and $u_2 \in p\ker\sigma(\Pi_+)(\eta)$. Using this fact and (18), it is straightforward to show that $u_1 = 0$, $v_1 = 0$, $u_2 = 0$, $v_2 = 0$.

The proof of Proposition 3.3 is complete. \square

Lemma 3.4. *If $\mathcal{P} = (0, P)$ is a projection, then the operators*

$$D_M \otimes 1_{\text{Im } \mathcal{P}} \quad \text{and} \quad D_X^* \otimes 1_{\text{Im } P}$$

are stably homotopic. In particular,

$$\text{ind } D_M \otimes 1_{\mathcal{P}} = -\text{ind}(D_X \otimes 1_{\text{Im } P}).$$

Remark 3.5. In this lemma, the range of the projection P is the space of sections of a finite-dimensional vector bundle over X . Thus, we can use this bundle to twist the Dirac operator on X , and the index is well defined.

Proof. Since the first component of \mathcal{P} is zero, we see that the boundary value problem is reduced to the operator

$$\begin{pmatrix} 1 & \tilde{D}_X^* \\ (1 \otimes (1 - P))\tilde{D}_X & -1 \otimes (1 - P) \end{pmatrix} : \begin{array}{c} C^\infty(X, S_X^+ \otimes \text{Im}(1 - P)) \\ \oplus \\ C^\infty(X, S_X^- \otimes C^\infty(\Omega, S_\Omega \otimes \mathbb{C}^n)) \end{array} \xrightarrow{\quad} \begin{array}{c} C^\infty(X, S_X^+ \otimes C^\infty(\Omega, S_\Omega \otimes \mathbb{C}^n)) \\ \oplus \\ C^\infty(X, S_X^- \otimes \text{Im}(1 - P)) \end{array}$$

on the boundary. Modulo compact operators, this operator is just the direct sum of

$$\begin{pmatrix} 1 & [1 \otimes (1 - P)]\tilde{D}_X^* \\ (1 \otimes (1 - P))\tilde{D}_X & -1 \end{pmatrix} : \begin{array}{c} C^\infty(X, S_X^+ \otimes \text{Im}(1 - P)) \\ \oplus \\ C^\infty(X, S_X^- \otimes \text{Im}(1 - P)) \end{array} \xrightarrow{\quad} \begin{array}{c} C^\infty(X, S_X^+ \otimes \text{Im}(1 - P)) \\ \oplus \\ C^\infty(X, S_X^- \otimes \text{Im}(1 - P)) \end{array} \quad (19)$$

and

$$\begin{pmatrix} 0 & (1 \otimes P)\tilde{D}_X^* \\ 0 & 0 \end{pmatrix} : \begin{array}{c} 0 \\ \oplus \\ C^\infty(X, S_X^- \otimes \text{Im } P) \end{array} \xrightarrow{\quad} \begin{array}{c} C^\infty(X, S_X^+ \otimes \text{Im } P) \\ \oplus \\ 0 \end{array} \quad (20)$$

Here we have used the compactness of the commutators $[1 \otimes P, \tilde{D}_X]$ and $[1 \otimes P, \tilde{D}_X^*]$. (The commutators of the corresponding principal and operator-valued symbols in the sense of [6] are zero.)

The operator (19) is homotopic to $1 \oplus (-1)$ in the class of elliptic operators. (To define the homotopy, we multiply \tilde{D}_X^* and \tilde{D}_X by a parameter ε varying from 1 down to 0.) The principal symbol of (20) coincides on the cosphere bundle with the principal symbol of the Dirac operator D_X^* on X twisted by the finite-dimensional vector bundle $\text{Im } P$.

The proof of Lemma 3.4 is complete. \square

3. Classes in analytic K -homology of manifolds with edges. We now assign a class in the analytic K -homology of the algebra $C(\mathcal{M})$ of continuous functions on \mathcal{M} to the boundary value problem (13).

Suppose that the Sobolev exponent s is an integer ≥ 1 . First, we reduce (13) to a boundary value problem in L^2 spaces. To this end, consider the composition of (13) with appropriate powers of order reduction operators

$$T: H^s(M, E) \longrightarrow H^{s-1}(M, E), \quad s \geq 1, \quad E \in \text{Vect}(M). \quad (21)$$

(An explicit construction of an elliptic operator (21) that has index zero and does not require boundary conditions is given, e.g., in [13, Prop. 20.3.1].)

This composition is given by the Fredholm operator

$$\begin{array}{ccc}
pL^2(M, S_M^+ \otimes \mathbb{C}^n) & & pL^2(M, S_M^- \otimes \mathbb{C}^n) \\
\oplus & & \oplus \\
D_M \widetilde{\otimes} 1_{\text{Im } \mathcal{P}}: (1-P)L^2(\partial M, \pi^* S_X^+ \otimes S_\Omega \otimes \mathbb{C}^n) & \longrightarrow & L^2(\partial M, \pi^* S_X^+ \otimes S_\Omega \otimes \mathbb{C}^n) \\
\oplus & & \oplus \\
(1-p)L^2(\partial M, \pi^* S_X^- \otimes S_\Omega \otimes \mathbb{C}^n) & & (1-P)L^2(\partial M, \pi^* S_X^- \otimes S_\Omega \otimes \mathbb{C}^n)
\end{array}$$

where

$$D_M \widetilde{\otimes} 1_{\text{Im } \mathcal{P}} = T^{s-1}(D_M \otimes 1_{\text{Im } \mathcal{P}})T^{-s}.$$

This bounded Fredholm operator acts in subspaces of L^2 spaces, which can be equipped with $C(\mathcal{M})$ -module structures. (A function $f \in C(\mathcal{M})$ acts in L^2 -spaces on M and ∂M as the multiplication by f and $f|_{\partial M}$, respectively.) Moreover, the operator commutes with the $C(\mathcal{M})$ -module structure modulo compact operators. (This is clear for smooth functions and follows by continuity for continuous functions.)

Thus, our operator is an abstract elliptic operator in the sense of Atiyah on \mathcal{M} [14]. Hence a standard construction (e.g., see [4]) defines the corresponding class in analytic K -homology of \mathcal{M} , which we denote by

$$[D_M \otimes 1_{\text{Im } \mathcal{P}}] \in K^0(C(\mathcal{M})) \equiv K_0(\mathcal{M}). \quad (22)$$

4 Poincaré isomorphism

1. Main theorem.

Theorem 4.1. *The mapping*

$$\begin{aligned}
\tau_{\mathcal{M}}: K_0(\mathcal{A}) &\longrightarrow K^0(C(\mathcal{M})), \\
\mathcal{P} &\longmapsto [D_M \otimes 1_{\text{Im } \mathcal{P}}],
\end{aligned} \quad (23)$$

is an isomorphism.

Proof. 0. It is easy to verify that the mapping $\mathcal{P} \mapsto [D_M \otimes 1_{\text{Im } \mathcal{P}}]$ induces a well-defined homomorphism of the K -group into the K -homology group. (Indeed, zero projections are taken to trivial operators by Lemma 3.4, and homotopic projections are taken to homotopic operators.)

1. On the one hand, the exact sequence

$$\begin{array}{ccccccc}
0 \rightarrow & C(X, \mathcal{K}) & \longrightarrow & \overline{\mathcal{A}} & \longrightarrow & C(M) & \rightarrow 0 \\
& F & \longmapsto & (0, F) & & & \\
& & & (f, F) & \longmapsto & f & \\
& & & & & &
\end{array}$$

of C^* -algebras, where the ideal consists of compact Toeplitz symbols and the quotient consists of continuous functions on M , induces the exact sequence

$$\dots K_1(C(M)) \xrightarrow{\delta_1} K_0(C(X, \mathcal{K})) \rightarrow K_0(\overline{\mathcal{A}}) \rightarrow K_0(C(M)) \xrightarrow{\delta_0} K_1(C(X, \mathcal{K})) \dots \quad (24)$$

in K -theory. By definition, the boundary mapping δ_0 takes the class of each vector bundle E on M (the range of a projection over $C(M)$) to the index

$$\text{ind}(D_\Omega \otimes 1_{E|_{\partial M}}) \in K^1(X) \simeq K_1(C(X, \mathcal{K}))$$

of the family of Dirac operators in the fibers of π twisted by the restriction of E to ∂M .

2. On the other hand, the short exact sequence

$$0 \rightarrow C_0(\mathcal{M} \setminus X) \longrightarrow C(\mathcal{M}) \longrightarrow C(X) \rightarrow 0$$

of function algebras induces the sequence

$$\cdots K^1(C_0(\mathcal{M} \setminus X)) \rightarrow K^0(C(X)) \rightarrow K^0(C(\mathcal{M})) \rightarrow K^0(C_0(\mathcal{M} \setminus X)) \rightarrow K^1(C(X)) \cdots \quad (25)$$

in K -homology.

3. We combine the sequences (24) and (25) in the diagram

$$\begin{array}{ccccccc} K_1(C(M)) & \rightarrow & K_0(C(X, \mathcal{K})) & \rightarrow & K_0(\overline{\mathcal{A}}) & \rightarrow & K_0(C(M)) \rightarrow K_1(C(X, \mathcal{K})) \\ \tau_M \downarrow & & \downarrow \tau_X & & \downarrow \tau_M & & \downarrow \tau_X \\ K^1(C_0(\mathcal{M} \setminus X)) & \rightarrow & K^0(C(X)) & \rightarrow & K^0(C(\mathcal{M})) & \rightarrow & K^0(C_0(\mathcal{M} \setminus X)) \rightarrow K^1(C(X)) \end{array} \quad (26)$$

Here τ_X and τ_M are the Poincaré isomorphisms on X and M . (Poincaré isomorphisms on smooth manifolds and manifolds with boundary are considered, e.g., in [15, 16].) More precisely, the mappings of the even K -groups are defined in terms of Dirac operators twisted by corresponding vector bundles. The definition of τ_X and τ_M on the odd K -groups can be obtained by a suspension argument.

4. Let us show that the diagram (26) commutes neglecting sign.

First, let us prove that the square

$$\begin{array}{ccc} K_0(C(M)) & \xrightarrow{\delta_0} & K_1(C(X, \mathcal{K})) \\ \tau_M \downarrow & & \downarrow \tau_X \\ K^0(C_0(\mathcal{M} \setminus X)) & \rightarrow & K^1(C(X)) \end{array} \quad (27)$$

commutes. Indeed, let E be a vector bundle on M ; then the composition of mappings passing through the top right corner in (27) takes the class $[E] \in K_0(C(M))$ to

$$[D_X] \cdot \text{ind}(D_\Omega \otimes 1_{E|_{\partial M}}) \in K^1(C(X)),$$

i.e., to the class of the Dirac operator on X twisted by $\text{ind}(D_\Omega \otimes 1_{E|_{\partial M}}) \in K_1(C(X))$. If we apply the composition of mappings passing through the bottom left corner of the square to $[E]$, then we obtain the element

$$\pi_*[D_{\partial M} \otimes 1_{E|_{\partial M}}] \in K^1(C(X)).$$

Lemma 4.2. *One has*

$$\pi_*[D_{\partial M} \otimes 1_{E|_{\partial M}}] = [D_X] \cdot \text{ind}(D_\Omega \otimes 1_{E|_{\partial M}}) \in K^1(C(X)). \quad (28)$$

Proof (see [4]). Consider the zero-order pseudodifferential operator $\mathcal{D} = \Delta_{\partial M}^{-1/2} \circ (D_{\partial M} \otimes 1_{E|_{\partial M}})$ on the total space ∂M of π as a pseudodifferential operator on X with operator-valued symbol in the sense of [17]. Then by the generalized Luke formula [4] we obtain

$$\pi_*[\mathcal{D}] = Q[\text{ind } \sigma_L(\mathcal{D})] \in K^1(C(X)),$$

where $\sigma_L(\mathcal{D})$ is a symbol of compact fiber variation on T^*X , $\text{ind } \sigma_L(\mathcal{D}) \in K_1(C_0(T^*X))$ is its index, and

$$Q: K_1(C_0(T^*X)) \longrightarrow K^1(C(X))$$

is the mapping, induced by quantization, that takes the class of each elliptic symbol to the class of the corresponding operator in K -homology.

The symbol of \mathcal{D} in the sense of [17] is (up to the invertible factor $(\xi^2 + \Delta_\Omega)^{1/2}$ defined by the Laplacian $\Delta_{\partial M}$) the exterior tensor product

$$\sigma(D_X)(x, \xi) \# (D_{\Omega_x} \otimes 1_{E|_{\Omega_x}}).$$

Hence we obtain

$$\text{ind } \sigma_L(\mathcal{D}) = [\sigma(D_X)] \text{ind}(D_\Omega \otimes 1_{E|_{\partial M}})$$

by the multiplicative property of the index. An application of Q gives the desired relation (28).

The proof of the lemma is complete. \square

Thus, the square (27) commutes. By the suspension argument, one proves that the same is true of the leftmost square in (26).

Let us prove the commutativity of the square

$$\begin{array}{ccc} K_0(C(X)) & \rightarrow & K_0(\overline{\mathcal{A}}) \\ \tau_X \downarrow & & \downarrow \tau_{\mathcal{M}} \\ K^0(C(X)) & \rightarrow & K^0(C(\mathcal{M})). \end{array} \quad (29)$$

Let $[P] \in K_0(C(X))$ be the class of a vector bundle $\text{Im } P$ on X . The composition of mappings passing through the top left corner of (29) takes $[P]$ to the K -homology class of the operator

$$D_X \otimes 1_{\text{Im } P}.$$

On the other hand, the composition of mappings in the opposite order takes this element (by Lemma 3.4) to the K -homology class of the operator

$$D_X^* \otimes 1_{\text{Im } P}.$$

It is clear that these two classes are equal neglecting sign.

The square

$$\begin{array}{ccc} K_0(\overline{\mathcal{A}}) & \rightarrow & K_0(C(M)) \\ \tau_{\mathcal{M}} \downarrow & & \downarrow \tau_M \\ K^0(C(\mathcal{M})) & \rightarrow & K^0(C_0(\mathcal{M} \setminus X)) \end{array} \quad (30)$$

commutes for obvious reasons. (The horizontal mappings are forgetful mappings.)

Thus, diagram (26) commutes (neglecting sign).

5. By applying the five lemma to diagram (26), we complete the proof of Theorem 4.1. \square

2. Obstruction to Fredholm realizations of Dirac operators on manifolds with edges. Let D_M be a Dirac operator on M associated with some spin^c -structure on M that induces spin^c -structures on the base and fibers of $\pi: \partial M \longrightarrow X$. Let D_X and D_Ω be the Dirac operator on the base and the family of Dirac operators on the fibers, respectively.

The operator D_M defines a class

$$[D_M] \in K^{\dim M}(C_0(\mathcal{M} \setminus X))$$

in K -homology. Consider the inclusion $j: C_0(\mathcal{M} \setminus X) \rightarrow C(\mathcal{M})$.

Proposition 4.3. *The element $[D_M] \in K^{\dim M}(C_0(\mathcal{M} \setminus X))$ is in the range of the mapping*

$$j^*: K^*(C(\mathcal{M})) \longrightarrow K^*(C_0(\mathcal{M} \setminus X))$$

of analytic K -homology groups if and only if Assumption 3.1 is satisfied, i.e., if and only if

$$\text{ind } D_\Omega = 0.$$

Proof. The K -homology exact sequence induced by j has the form

$$\cdots K^{*+1}(C_0(\mathcal{M} \setminus X)) \rightarrow K^*(C(X)) \rightarrow K^*(C(\mathcal{M})) \xrightarrow{j^*} K^*(C_0(\mathcal{M} \setminus X)) \xrightarrow{\partial} K^{*+1}(C(X)) \cdots \quad (31)$$

By exactness, we have $[D_M] = j^*x$ for some $x \in K^*(C(\mathcal{M}))$ if and only if $\partial[D_M] = 0$. The element $\partial[D_M]$ is equal to

$$\partial[D_M] = [D_X] \cdot \text{ind } D_\Omega \in K^*(C(X))$$

(see (28)). Since $[D_X]$ is a generator of $K^*(C(X))$ as a free $K_*(C(X))$ -module, we conclude that $\partial[D_M] = 0$ is equivalent to $\text{ind } D_\Omega = 0$.

The proof of proposition is complete. \square

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